Thrust Equation for a Turbojet
Single Inlet/Outlet

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April 7, 2017

1 Introduction

In this document, we will derive the thrust equation for a single inlet/single outlet engine. I use the term "turbojet" in the title as more of a buzzword than anything, because the derived equation will hold for any engine with a single inlet and single outlet.

Mass Flow Rate

Mass flow rates will be used throughout this derivation, so it's important to understand what the term means, as well as the units associated with it. Mass flow rates are useful for evaluating general thrust of engines because it is a number that can be easily found for pretty much any engine on the internet. The mass flow rate of a fluid is defined as the amount mass of fluid flowing per unit time. The 1D equation for mass flow rate is given in Eq. (1).

$$\dot{m} = \rho u A \tag{1}$$

To see how the units work out for this equation, see Eq. (2) below.

$$\left[\frac{kg}{s}\right] = \left[\frac{kg}{m^3}\right]\left[\frac{m}{s}\right]\left[m^2\right] \tag{2}$$

Velocity

In a general three-dimensional flow, we have three velocity components, one for each coordinate direction. We can write the general formula for the velocity at any given point as seen in Eq. (3).

$$\vec{V} = u\hat{\imath} + v\hat{\jmath} + w\hat{k} \tag{3}$$

The goal of the analysis is to develop and expression for the thrust, which will be aligned with the X-direction. Thus, we can say that the Y-direction and Z-direction velocities are zero, and write the velocity vector as follows.

$$\vec{V} = u\hat{\imath} \tag{4}$$

Area

One of the most deceivingly confusing parts the conservation equations is the control volume surface area term, and what it means. For a blog post regarding this topic, visit the following website.

http://www.joshtheengineer.com/2017/01/02/surface-double-integrals/

To sum it all up, the surface area terms used in the conservation equations are shown below.

$$\begin{aligned} d\vec{S_1} &= \hat{n}_1 dA_1 \\ d\vec{S_2} &= \hat{n}_2 dA_2 \end{aligned} \tag{5}$$

This means that after integration based on the outward normal vectors, we will have the following.

$$\begin{aligned} S_1 &= -A_1 \\ S_2 &= +A_2 \end{aligned} \tag{6}$$

2 Engine Control Volume Schematic

A schematic of the engine and its surrounding control volume can be seen in Fig. 1. The engine is drawn in red to distinguish it from the control volume and associated variables. Flow comes in from the left and exits to the right.

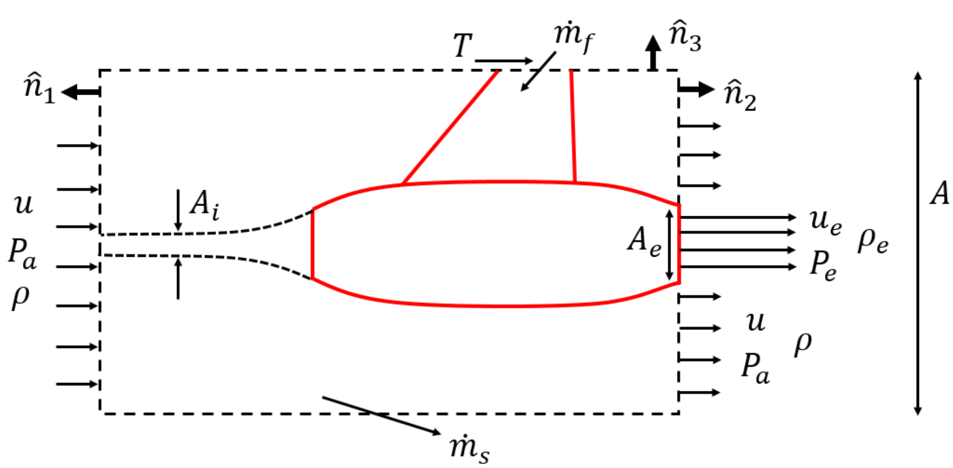

Figure 1: Engine control volume schematic.

3 Conservation of Mass

Here is the mass conservation in integral form.

$$\iint \rho \vec{V} \cdot d\vec{S} = 0 \tag{7}$$

We can first expand out the terms, using the velocity vector defined previously (only X-direction) and the control surface normal vectors.

$$\rho \left[(u\hat{\imath}) \cdot (-A_i\hat{\imath})\right] + \rho \left[(u\hat{\imath}) \cdot \big(-(A - A_i)\big)\hat{\imath}\right] + \rho_e \left[(u_e\hat{\imath}) \cdot (A_e\hat{\imath})\right] + \rho \left[(u\hat{\imath}) \cdot \big(A - A_e\big)\hat{\imath}\right] - \dot{m}_f + \dot{m}_s = 0 \tag{8}$$

Multiply the terms through, and then simplify using the definition of the mass flow rate shown in Eq. (1).

$$-\underbrace{\rho u A_i}_{\dot{m}_a} - \rho u \left(A - A_i\right) + \underbrace{\rho_e u_e A_e}_{\dot{m}_e} + \rho u \left(A - A_e\right) - \dot{m}_f + \dot{m}_s = 0 \tag{9}$$

$$-\dot{m}_a - \rho u(A - A_i) + \dot{m}_e + \rho u(A - A_e) - \dot{m}_f + \dot{m}_s = 0 \tag{10}$$

Solve for the $\dot{m}_s$ term. We will use this term as a sort of bridge between the conservation of mass and the conservation of momentum.

$$\begin{aligned} \dot{m}_s &= \dot{m}_a + \dot{m}_f + \rho u(A - A_i) - \rho u(A - A_e) - \dot{m}_e \\ &= \dot{m}_a + \dot{m}_f + \rho u A - \rho u A_i - \rho u A + \rho u A_e - \dot{m}_e \\ &= \dot{m}_a + \dot{m}_f - \rho u A_i + \rho u A_e - \dot{m}_e \end{aligned} \tag{11}$$

$$\boxed{\dot{m}_s = \dot{m}_a + \dot{m}_f + \rho u(A_e - A_i) - \dot{m}_e} \tag{12}$$

4 Conservation of Momentum

Here is the mass conservation in integral form. We will assume the momentum from the fuel is negligible.

$$\iint_S \rho \vec{V} \left(\vec{V} \cdot d\vec{S}\right) + \iint_S P d\vec{S} = 0 \tag{13}$$

Expand out the terms as we did previously with the mass conservation equation.

$$\rho\left(u\hat{\imath}\right)\left[(u\hat{\imath}) \cdot (\text{-}A_i\hat{\imath})\right] + \rho\left(u\hat{\imath}\right)\left[(u\hat{\imath}) \cdot \big(\text{-}\left(A - A_i\right)\big)\hat{\imath}\right] + \rho_e\left(u_e\hat{\imath}\right)\left[(u_e\hat{\imath}) \cdot (A_e\hat{\imath})\right] + \rho\left(u\hat{\imath}\right)\left[(u\hat{\imath}) \cdot \big(A - A_e\big)\hat{\imath}\right] + \dot{m}_s u + P_a\left(\text{-}A_i\right) + P_a\left[\text{-}\left(A - A_i\right)\right] + P_e A_e + P_a\left[A - A_e\right] = T \tag{14}$$

Simplify the equation by multiplying terms through, making sure to keep track of the signs of the terms.

$$\begin{aligned} &\text{-}\rho u A_i \left(u\right) - \rho u^2 \left(A - A_i\right) + \rho_e u_e A_e \left(u_e\right) + \rho u^2 \left(A - A_e\right) \\ &\qquad + \dot{m}_s u - P_a A_i - P_a \left(A - A_i\right) + P_e A_e + P_a \left(A - A_e\right) = T \end{aligned} \tag{15}$$

Keep simplifying using the following equations for the mass flow rates.

$$\begin{aligned} \dot{m}_a &= \rho u A_i \\ \dot{m}_e &= \rho_e u_e A_e \end{aligned} \tag{16}$$

$$\begin{aligned} &\text{-}\dot{m}_a u + \dot{m}_e u_e + \rho u^2 \left(A_i - A_e\right) + \dot{m}_s \\ &\qquad - \cancel{P_a A_i} - \cancel{P_a A} + \cancel{P_a A_i} + P_e A_e + \cancel{P_a A} - P_a A_e = T \end{aligned} \tag{17}$$

Solve for the thrust, noting that the $\dot{m}_s$ term appears on the right-hand side of the equation.

$$\boxed{T = \text{-}\dot{m}_a u + \dot{m}_e u_e + \rho u^2 \left(A_i - A_e\right) + \dot{m}_s u + A_e \left(P_e - P_a\right)} \tag{18}$$

5 Combining Mass and Momentum

Recall the two boxed expressions from before, repeated below in Eqs. (19) and (20).

$$\boxed{\dot{m}_s = \dot{m}_a + \dot{m}_f + \rho u (A_e - A_i) - \dot{m}_e} \tag{19}$$

$$\boxed{T = \text{-}\dot{m}_a u + \dot{m}_e u_e + \rho u^2 \left(A_i - A_e\right) + \dot{m}_s u + A_e \left(P_e - P_a\right)} \tag{20}$$

Substitute the first expression into the second expression.

$$\begin{aligned} T = \text{-}\dot{m}_a u &+ \dot{m}_e u_e + \rho u^2 \left(A_i - A_e\right) + A_e \left(P_e - P_a\right) \\ &\qquad + u \left[\dot{m}_a + \dot{m}_f + \rho u (A_e - A_i) - \dot{m}_e\right] \end{aligned} \tag{21}$$

Distribute the velocity u through the last term, and then cancel like terms.

$$\begin{aligned} T = \text{-}\dot{m}_a u &+ \dot{m}_e u_e + \cancel{\rho u^2 \left(A_i - A_e\right)} + A_e \left(P_e - P_a\right) \\ &\qquad + \dot{m}_a u + \dot{m}_f u + \cancel{\rho u^2 \left(A_e - A_i\right)} - \dot{m}_e u \end{aligned} \tag{22}$$

The air entering the engine mixes with the injected fuel, so the mass flow rate exiting the engine is the sum of those two, as seen below.

$$\dot{m}_a u + \dot{m}_f u = u \underbrace{\left(\dot{m}_a + \dot{m}_f\right)}_{\dot{m}_e} = \dot{m}_e u \tag{23}$$

Note how the terms now cancel in the equation.

$$T = \text{-}\dot{m}_a u + \dot{m}_e u_e + A_e \left(P_e - P_a\right) + \cancel{\dot{m}_e u} - \cancel{\dot{m}_e u} \tag{24}$$

Now plug in the exit mass flow rate in terms of the air and fuel mass flow rates.

$$T = \text{-}\dot{m}_a u + \left(\dot{m}_a + \dot{m}_f\right) u_e + A_e \left(P_e - P_a\right) \tag{25}$$

Distribute the exit velocity through the second term in the above equation.

$$T = \text{-}\dot{m}_a u + \dot{m}_a u_e + \dot{m}_f u_e + A_e \left(P_e - P_a\right) \tag{26}$$

Combine the terms with $\dot{m}_a$ and multiply the fuel mass flow rate term by $\dot{m}_a/\dot{m}_a$.

$$T = \dot{m}_a \left(u_e - u\right) + \frac{\dot{m}_f}{\dot{m}_a} \dot{m}_a u_e + A_e \left(P_e - P_a\right) \tag{27}$$

Note that the fuel-to-air ratio, f, is $\dot{m}_f/\dot{m}_a$.

$$T = \dot{m}_a \left(u_e - u\right) + f \dot{m}_a u_e + A_e \left(P_e - P_a\right) \tag{28}$$

Bring the $\dot{m}_a$ term into the combined term.

$$T = \dot{m}_a \left(u_e - u + f u_e\right) + A_e \left(P_e - P_a\right) \tag{29}$$

Finally, combine the u_e term in the parentheses to give the final thrust equation for a single inlet, single outlet engine.

$$\boxed{T = \dot{m}_a \left[u_e \left(1 + f\right) - u\right] + A_e \left(P_e - P_a\right)} \tag{30}$$

6 Thrust Equation Simplifications

If the nozzle is not choked, then the exit pressure of the engine will equal the atmospheric pressure. The topic of nozzle choking will be discussing in a different post. If the nozzle is not choked, then the second term in the thrust equation disappears because $P_e = P_a$.

$$\boxed{T = \dot{m}_a \left[u_e \left(1 + f\right) - u\right]} \tag{31}$$

In typical engines, the fuel-to-air ratio, f, is very small. If we assume that $f \ll 1$, then we can write the thrust equation as follows. This is still assuming that our nozzle is not choked.

$$\boxed{T = \dot{m}_a \left[u_e - u\right]} \tag{32}$$

This simplified expression shows that the thrust will increase if we increase the air flow rate through the engine or if we increase the exit velocity of the engine. It also shows that increasing flight speed u will decrease the thrust.