Integration
U-Substitution

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In this document, I will be going through how to solve an integral using *u-substitution.* The integral we are going to solve is shown in Eq. (1) below.

$$\int x \sqrt{1 - x^2} \, dx \tag{1}$$

As the name suggests, we will be defining a variable, u, and substituting it into the integral to hopefully make it easier to solve. Let's define u as shown in Eq. (2).

$$u = 1 - x^2 \tag{2}$$

We will also need its derivative, du.

$$du = -2x dx \tag{3}$$

Note that we can plug Eq. (2) directly into Eq. (1), but that we need to rearrange Eq. (3) to solve for $x dx$ before we can plug it into (1).

$$-\frac{1}{2} du = x dx \tag{4}$$

Let's now plug in Eq. (2) and Eq. (4) into Eq. (1) so that we can rewrite the integral in terms of u.

$$\int x \sqrt{1 - x^2} \, dx = \int -\frac{1}{2} \sqrt{u} \, du = \frac{-1}{2} \int u^{\frac{1}{2}} \, du \tag{5}$$

We can easily integrate this expression, as seen below.

$$\frac{-1}{2} \int u^{\frac{1}{2}} \, du = \frac{-1}{2} \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}} \right] = \frac{-1}{2} \left(\frac{2}{3} \right) u^{\frac{3}{2}} = \frac{-u^{\frac{3}{2}}}{3} \tag{6}$$

Recall from Eq. (2) how we defined u in terms of x. We can plug Eq. (2) into Eq. (6) to get the final solution to this integral.

$$\boxed{\int x \sqrt{1 - x^2} \, dx = \frac{-\left(1 - x^2\right)^{\frac{3}{2}}}{3}} \tag{7}$$